New Representations of Bertrand Pairs in Euclidean 3-Space

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Abstract

In this work, we studied the properties of the spherical indicatrices of a Bertrand curve and its mate curve and presented some characteristic properties in the cases that Bertrand curve and its mate curve are slant helices, spherical indicatrices are slant helices and we also researched that whether the spherical indicatrices made new curve pairs in the means of Mannheim, involte-evolute and Bertrand pairs. Further more, we investigated the relations between the spherical images and introduced new representations of spherical indicatrices.

Key words: Bertrand curve, Bertrand mate curve, Spherical indicatrix, Slant helix.
AMS Subject Classification 2000: 53A04

1 Introduction

It is well known that the specific curve pairs are the most popular subjects in curve and surface theory and Bertrand curve is one of them. A Bertrand curve is a curve in Euclidean 3-space whose principal normal is the principal normal of another curve. We can see in most textbooks, a characteristic property of the Bertrand curve which asserts the existence of a linear relation between curvature and torsion. The characteristic property-linear relation-is deduced as an application of the Frenet-Serret formulae.

In this study, the spherical indicatrices of a Bertrand curve and its mate curve are given. In order to make a Bertrand curve and its mate curve slant helix, the feature that spherical indicatrices curve's need to have are examined. It

Preprint submitted to Elsevier *30 December 2011*

is seen that the curves of the spherical indicatrices do not form any specific curve pairs.

Let $\gamma : I \longrightarrow IR^3$ be a curve with $\gamma'(s) \neq 0$, where $\gamma'(s) = d\,\gamma(s)/ds$. The arc-lenght s of a curve $\gamma(s)$ is determined such that $\|\gamma'(s)\| = 1$. This shows us that $T(s) = \gamma'(s)$ and $T(s)$ is called a tangent vector of γ at $\gamma(s)$. We define the curvature of γ by $\kappa(s) = \|\gamma''(s)\|$. If $\kappa(s) \neq 0$, the unit principal normal vector $N(s)$ of the curve at $\gamma(s)$ is given by $\gamma''(s) = \kappa(s)\,N(s)$. The unit vector $B(s) = T(s)\,\Lambda N(s)$ is called the unit binormal vector of γ at $\gamma(s)$. Then we have the Frenet-Serret formulae

$$T' = \kappa N \ , \ \ N' = -\kappa T + \tau B \ , \ \ B' = -\tau N$$

where $\tau(s)$ is the torsion of γ at $\gamma(s)$ [5].

The curve γ with nonzero curvature is called a Bertrand curve if there exists a curve $\widetilde{\gamma} : I \longrightarrow IR^3$ such that the principal normal vectors of γ and $\widetilde{\gamma}$ are linearly dependent at the corresponding points for each $s \in I \subset IR$. In this case, $\widetilde{\gamma}$ is called a Bertrand mate of γ and there exists a relationship between the position vectors as

$$\widetilde{\gamma}(s^*) = \gamma(s) + \lambda N(s)$$

and we can write $N = \epsilon \widetilde{N}$ where $\epsilon = \pm 1$ and λ is the distance between the curves γ and $\widetilde{\gamma}$ at the corresponding points for each s. The pair of $(\gamma, \widetilde{\gamma})$ is called a Bertrand pair. λ is a constant for Bertrand pairs[3,6].

The curve $\widetilde{\gamma}$ is called involute of γ if the tangent vectors are orthogonal at the corresponding points for each $s \in I \subset IR$. In this case, γ is called evolute of the curve $\widetilde{\gamma}$ and there exists a relationship between the position vectors as

$$\widetilde{\gamma}(s^*) = \gamma(s) + \lambda T(s)$$

where λ is the distance between the curves γ and $\widetilde{\gamma}$ at the corresponding points for each s.The pair of $(\gamma, \widetilde{\gamma})$ is called a involute-evolute pair. λ is not a constant for involute-evolute pairs[5].

The curve $\widetilde{\gamma}$ is called Mannheim curve if there exists a corresponding relationship between the space curves γ and $\widetilde{\gamma}$ such that the principal normal lines of $\widetilde{\gamma}$ coincide with the binormal lines of γ at the corresponding points of the curves. In this case, γ is called as a Mannheim partner curve of $\widetilde{\gamma}$ and there exists a relationship between the position vectors as

$$\widetilde{\gamma}(s^*) = \gamma(s) + \lambda B(s)$$

where λ is the distance between the curves γ and $\widetilde{\gamma}$ at the corresponding points for each s. The pair of $(\gamma, \widetilde{\gamma})$ is called a Mannheim pair. λ is a constant for Mannheim pairs[1,2].

On the other hand, Izumiya and Takeuchi have introduced the concept of slant helix by saying that the normal lines make a constant angle with a fixed straight line. They characterize a slant helix if and only if the geodesic curvature of the principal image of the principal normal indicatrix

$$\Gamma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

is a constant function[4,6] .

In this study, we denote that T, N, B, κ, τ and $\widetilde{T}$, $\widetilde{N}$, $\widetilde{B}$, $\widetilde{\kappa}$, $\widetilde{\tau}$ are the Frenet equipments of γ and $\widetilde{\gamma}$, respectively. Tangent, principal normal and binormal vectors are described for the spherical curves which are called tangent, principal normal and binormal indicatrices for both the curves γ and $\widetilde{\gamma}$, respectively. Throughout this study, the Bertrand curve and its mate curve are accepted as non helix and non planar.

2 Spherical indicatrices of a Bertrand Curve

In this section, we introduced the spherical indicatrices of a Bertrand curve according to its mate curve in Euclidean 3-space and gave considerable results by using the properties of the curves. Let γ be a Bertrand curve with a Bertrand mate curve $\widetilde{\gamma}$, then

$$\gamma(s) = \widetilde{\gamma}(s^*) - \lambda\epsilon\widetilde{N} \tag{1}$$

where λ is

$$\lambda = \frac{-\epsilon\widetilde{g}}{\widetilde{\kappa}\left(\widetilde{g}-\widetilde{f}\right)} \tag{2}$$

and

$$\widetilde{f} = \frac{\widetilde{\tau}}{\widetilde{\kappa}}, \quad \widetilde{g} = \frac{\widetilde{\tau}'}{\widetilde{\kappa}'}.$$

Theorem 1 *Let γ be a Bertrand curve, then we have Frenet formula:*

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N$$

where

$$T = \frac{-1}{\sqrt{1+\widetilde{g}^2}}\left\{\widetilde{T} - \widetilde{g}\widetilde{B}\right\}, \quad N = \epsilon\widetilde{N}, \quad B = \frac{-\epsilon}{\sqrt{1+\widetilde{g}^2}}\left\{\widetilde{g}\widetilde{T} + \widetilde{B}\right\} \tag{3}$$

and

$$s=\int\frac{\widetilde{f}\sqrt{1+\widetilde{g}^{2}}}{\widetilde{g}-\widetilde{f}}ds^{*},\quad \kappa=\frac{-\epsilon\widetilde{\kappa}\left(1+\widetilde{f}\widetilde{g}\right)\left(\widetilde{g}-\widetilde{f}\right)}{\widetilde{f}\left(1+\widetilde{g}^{2}\right)},\quad \tau=\frac{\widetilde{\kappa}\left(\widetilde{g}-\widetilde{f}\right)^{2}}{\widetilde{f}\left(1+\widetilde{g}^{2}\right)} \tag{4}$$

The geodesic curvature of the principal image of the principal normal indicatrix of γ is

$$\Gamma=\frac{-\widetilde{\kappa}'\left(\widetilde{g}-\widetilde{f}\right)}{\widetilde{\kappa}^{2}\left(1+\widetilde{f}^{2}\right)^{3/2}} \tag{5}$$

Theorem 2 *A Bertrand curve is a slant helix if and only if its mate curve is a slant helix.*

PROOF. Let Γ and $\widetilde{\Gamma}$ be the geodesic curvatures of the principal normal curves of γ and $\widetilde{\gamma}$, respectively. Then $\widetilde{\Gamma}$ is

$$\widetilde{\Gamma}=\frac{\widetilde{\kappa}'\left(\widetilde{g}-\widetilde{f}\right)}{\widetilde{\kappa}^{2}\left(1+\widetilde{f}^{2}\right)^{3/2}}$$

and from (5) we get

$$\Gamma=-\widetilde{\Gamma}$$

So, it is easy to see that γ is a slant helix if and only if $\widetilde{\gamma}$ is a slant helix.

Let $p_1=\frac{1}{\sqrt{1+\widetilde{g}^2}}$ and $p_2=\frac{\widetilde{g}}{\sqrt{1+\widetilde{g}^2}}$. If we differentiate the equation (1) with respect to s^*, we get

$$\gamma'(s)\frac{ds}{ds^{*}}=\frac{-\widetilde{f}}{\widetilde{g}-\widetilde{f}}\widetilde{T}+\frac{\widetilde{f}\widetilde{g}}{\widetilde{g}-\widetilde{f}}\widetilde{B}.$$

Since

$$\frac{ds^{*}}{ds}=\frac{\widetilde{g}-\widetilde{f}}{\widetilde{f}\sqrt{1+\widetilde{g}^{2}}},$$

we can write

$$\gamma'(s)=p_1\widetilde{T}+p_2\widetilde{B}$$

again, if we differentiate the last equation with respect to s^* and put λ from equation (2), we obtain

$$\gamma''(s)=-p_1'\frac{ds^{*}}{ds}\widetilde{T}-\left\{p_1\widetilde{\kappa}+p_2\widetilde{\tau}\right\}\frac{ds^{*}}{ds}\widetilde{N}+p_2'\frac{ds^{*}}{ds}\widetilde{B}.$$

Since $\gamma''(s)$ has no components on $\widetilde{T}$ and $\widetilde{B}$, $p_1'=0$ and $p_2'=0$. Hence, p_1 and p_2 are constants. Thus, we prove the following theorem.

Theorem 3 *Let γ and $\widetilde{\gamma}$ be regular curves in E^3.If γ is a Bertrand curve with its mate curve $\widetilde{\gamma}$, $\widetilde{g}$ is a constant.*

Let γ be a unit speed regular curve in Euclidean 3-space with Frenet vectors T, N and B. The unit tangent vectors along the curve $\gamma(s)$ generates a curve $\gamma_t = T$ on the sphere of radius 1 about the origin. The curve γ_t is called the spherical indicatrix of T or more commonly, γ_t is called the tangent indicatrix of the curve γ. If $\gamma = \gamma(s)$ is a natural representations of the curve γ, $\gamma_t(s) = T(s)$ will be a representation of γ_t. Similarly, we can consider the principal normal indicatrix $\gamma_n = N(s)$ and the binormal indicatrix $\gamma_b = B(s)$.

The tangent indicatrix of a Bertrand curve is

$$\gamma_t = \frac{-1}{\sqrt{1+\widetilde{g}^2}}\left\{\widetilde{T} - \widetilde{g}\widetilde{B}\right\}.$$

Theorem 4 *If the Frenet frame of the tangent indicatrix $\gamma_t = T$ of the Bertrand curve is $\{T_t, N_t, B_t\}$, we have Frenet formula:*

$$T_t' = \kappa_t N_t, \quad N_t' = -\kappa_t T_t + \tau_t B_t, \quad B_t' = -\tau_t N_t$$

where

$$T_t = -\widetilde{N}, \quad N_t = \frac{1}{\sqrt{1+\widetilde{f}^2}}\left\{\widetilde{T} - \widetilde{f}\widetilde{B}\right\}, \quad B_t = \frac{1}{\sqrt{1+\widetilde{f}^2}}\left\{\widetilde{f}\widetilde{T} + \widetilde{B}\right\} \tag{6}$$

and

$$s_t = -\int \frac{\widetilde{\kappa}\left(\widetilde{g}-\widetilde{f}\right)^2}{\widetilde{f}\left(1+\widetilde{g}\right)^2}ds, \quad \kappa_t = \frac{\sqrt{1+\widetilde{g}^2}\sqrt{1+\widetilde{f}^2}}{\left(\widetilde{f}-\widetilde{g}\right)}, \quad \tau_t = \frac{-\widetilde{\kappa}'\sqrt{1+\widetilde{g}^2}}{\widetilde{\kappa}^2\left(1+\widetilde{f}^2\right)} \tag{7}$$

and s_t is a natural representation of the tangent indicatrix of the curve γ. κ_t and τ_t are the curvature and torsion of γ_t, respectively. The geodesic curvature of the principal image of the principal normal indicatrix of γ_t is

$$\Gamma_t = \frac{-\widetilde{\kappa}^3\left(1+\widetilde{f}^2\right)^{3/2}\left(\widetilde{g}-\widetilde{f}\right)^2\left\{\widetilde{\kappa}''\widetilde{\kappa}\left(1+\widetilde{f}^2\right) - 3\widetilde{\kappa}'^2\left(1+\widetilde{f}\widetilde{g}\right)\right\}}{\sqrt{1+\widetilde{g}^2}\left(\widetilde{\kappa}^4\left(1+\widetilde{f}^2\right)^3 + \widetilde{\kappa}'^2\left(\widetilde{f}-\widetilde{g}\right)^2\right)^{3/2}}\frac{ds^*}{ds_t}. \tag{8}$$

The arclengths s_t and s^* satisfy the equation

$$s_t = \int \frac{\widetilde{\kappa}\left(\widetilde{f}-\widetilde{g}\right)}{\sqrt{1+\widetilde{g}^2}}ds^*.$$

We can give the following theorem to characterize a Bertrand curve as a slant helix.

Theorem 5 *Let $(\gamma, \widetilde{\gamma})$ be a non-helical and non-planar Bertrand pair parameterized by the arclengths s and s^*. Thus, the followings are true.*

i. Bertrand curve is a slant helix if and only if the tangent indicatrix of the Bertrand curve is a spherical helix.

ii. Bertrand mate curve is a slant helix if and only if the tangent indicatrix of the Bertrand curve is a spherical helix.

Thus, we can state the following theorem to characterize the spherical image of tangent indicatrix of a Bertrand curve as a helix.

Theorem 6 *Let $(\gamma, \widetilde{\gamma})$ be a non-helical and non-planar Bertrand pair. Thus, the tangent indicatrix of the Bertrand curve is a spherical helix if and only if*

$$\widetilde{\kappa}''\widetilde{\kappa}\widetilde{f}^2 - 3\widetilde{\kappa}'^2\widetilde{g}\widetilde{f} + \widetilde{\kappa}''\widetilde{\kappa} - 3\widetilde{\kappa}'^2 = 0$$

is satisfied.

The principal normal indicatrix of the Bertrand curve is

$$\gamma_n = \epsilon\widetilde{N}.$$

Theorem 7 *If the Frenet frame of the principal normal indicatrix $\gamma_n = N$ of the Bertrand curve is $\{T_n, N_n, B_n\}$, we have Frenet formula:*

$$T_n' = \kappa_n N_n, \quad N_n' = -\kappa_n T_n + \tau_n B_n, \quad B_n' = -\tau_n N_n$$

where

$$\begin{aligned}
T_n &= \frac{-\epsilon}{\sqrt{1+\widetilde{f}^2}}\left\{\widetilde{T} - \widetilde{f}\widetilde{B}\right\} \\
N_n &= \frac{\epsilon}{\rho\sqrt{1+\widetilde{f}^2}}\left\{\widetilde{f}\widetilde{\kappa}'\left(\widetilde{g}-\widetilde{f}\right)\widetilde{T} - \widetilde{\kappa}^2\left(1+\widetilde{f}^2\right)^2\widetilde{N} + \widetilde{\kappa}'\left(\widetilde{g}-\widetilde{f}\right)\widetilde{B}\right\} \\
B_n &= \frac{1}{\rho}\left\{\widetilde{\kappa}^2\widetilde{f}\left(1+\widetilde{f}^2\right)\widetilde{T} + \widetilde{\kappa}'\left(\widetilde{g}-\widetilde{f}\right)\widetilde{N} + \widetilde{\kappa}^2\left(1+\widetilde{f}^2\right)\widetilde{B}\right\}
\end{aligned} \tag{9}$$

and

$$s_n = \int \frac{\widetilde{\kappa}\left(\widetilde{g}-\widetilde{f}\right)\sqrt{1+\widetilde{f}^2}}{\widetilde{f}\sqrt{1+\widetilde{g}^2}}ds, \quad \kappa_n = \frac{\rho}{\widetilde{\kappa}^2\left(1+\widetilde{f}^2\right)^{3/2}}$$

$$\tau_n = \frac{-\epsilon\left(\widetilde{g}-\widetilde{f}\right)}{\rho^2}\left\{\left(3\widetilde{\kappa}'^2 - \widetilde{\kappa}\widetilde{\kappa}''\right)\left(1+\widetilde{f}^2\right) + 3\widetilde{f}\widetilde{\kappa}'^2\left(\widetilde{g}-\widetilde{f}\right)\right\}$$

where

$$\rho = \sqrt{\widetilde{\kappa}'^2 \left(\widetilde{g} - \widetilde{f}\right)^2 + \widetilde{\kappa}^4 \left(1 + \widetilde{f}^2\right)^3}$$

and s_n *is a natural representation of the principal normal indicatrix of the curve* γ. κ_n *and* τ_n *are the curvature and torsion of* γ_n*, respectively.*

The arclengths s_n and s^* satisfy the equation

$$s_n = \int \widetilde{\kappa}\sqrt{1 + \widetilde{f}^2} ds^*.$$

We can give the following theorem to characterize the spherical image of the principal normal indicatrix of a Bertrand curve as a helix.

Theorem 8 *Let* $(\gamma, \widetilde{\gamma})$ *be a non-helical and non-planar Bertrand pair. Thus, the principal normal indicatrix of the Bertrand curve is planar if and only if*

$$\widetilde{\kappa}\widetilde{\kappa}'' \widetilde{f}^2 - 3\widetilde{\kappa}'^2 \widetilde{g}\widetilde{f} - \left(3\widetilde{\kappa}'^2 - \widetilde{\kappa}\widetilde{\kappa}''\right) = 0$$

is satisfied.

The binormal indicatrix of the Bertrand curve is

$$\gamma_b = \frac{-\epsilon}{\sqrt{1 + \widetilde{g}^2}} \left\{\widetilde{g}\widetilde{T} + \widetilde{B}\right\}.$$

Theorem 9 *If the Frenet frame of the binormal indicatrix* $\gamma_b = B$ *of the Bertrand curve is* $\{T_b, N_b, B_b\}$*, we have Frenet formula:*

$$T_b' = \kappa_b N_b, \quad N_b' = -\kappa_b T_b + \tau_b B_b, \quad B_b' = -\tau_b N_b$$

where

$$T_b = \epsilon \widetilde{N}, \quad N_b = \frac{-\epsilon}{\sqrt{1 + \widetilde{f}^2}} \left\{\widetilde{T} - \widetilde{f}\widetilde{B}\right\}, \quad B_b = \frac{1}{\sqrt{1 + \widetilde{f}^2}} \left\{\widetilde{f}\widetilde{T} + \widetilde{B}\right\} \tag{10}$$

and

$$s_b = -\int \frac{\widetilde{\kappa}\left(\widetilde{f} - \widetilde{g}\right)^2}{\widetilde{f}\left(1 + \widetilde{g}^2\right)} ds, \quad \kappa_b = \frac{\sqrt{1 + \widetilde{f}^2}\sqrt{1 + \widetilde{g}^2}}{\widetilde{f} - \widetilde{g}}, \quad \tau_b = \frac{-\epsilon\widetilde{\kappa}'\sqrt{1 + \widetilde{g}^2}}{\widetilde{\kappa}^2\left(1 + \widetilde{f}^2\right)} \tag{11}$$

and s_b *is a natural representation of the binormal indicatrix of the curve* γ *and it is equal to the total torsion of the curve* γ. κ_b *and* τ_b *are the curvature and torsion of* γ_b*, respectively. The geodesic curvature of the principal image of the principal normal indicatrix of* γ_b *is*

$$\Gamma_b = \frac{-\widetilde{\kappa}^3\left(1 + \widetilde{f}^2\right)^{3/2}\left(\widetilde{g} - \widetilde{f}\right)^2\left\{\widetilde{\kappa}''\widetilde{\kappa}\left(1 + \widetilde{f}^2\right) - 3\widetilde{\kappa}'^2\left(1 + \widetilde{f}\widetilde{g}\right)\right\}}{\sqrt{1 + \widetilde{g}^2}\left(\widetilde{\kappa}^4\left(1 + \widetilde{f}^2\right)^3 + \widetilde{\kappa}'^2\left(\widetilde{f} - \widetilde{g}\right)^2\right)^{3/2}} \frac{ds^*}{ds_b}. \tag{12}$$

From theorem 2 and from (7), (11), we conclude that

$$\Gamma = -\widetilde{\Gamma} = \frac{\tau_t}{\kappa_t} = \frac{\tau_b}{\kappa_b}$$

and $\Gamma_t = \Gamma_b$.Thus, we can give the following corollary.

Corollary 10 *The spherical images of the tangent and binormal indicatrices of a Bertrand curve are the curves with same curvature and same torsion.*

Corollary 11 *The arclengths s_b and s^* satisfy the following equation.*

$$s_b = \int \frac{\widetilde{\kappa}\left(\widetilde{f} - \widetilde{g}\right)}{\sqrt{1+\widetilde{g}^2}} ds^*$$

Since λ is a constant, from (2) we can write

$$\widetilde{\tau}' = -\epsilon\lambda\widetilde{\kappa}^2\widetilde{f}'$$

and since p_2 is a constant , we can write

$$\widetilde{\tau}' = c_1\widetilde{\kappa}'\sqrt{1+\widetilde{g}^2}.$$

Thus, we obtain

$$\frac{\widetilde{\kappa}^2\widetilde{f}'}{\widetilde{\kappa}'\sqrt{1+\widetilde{g}^2}} = \frac{-\epsilon c_1}{\lambda}$$

and we can also say that

$$\frac{\widetilde{\kappa}^2\widetilde{f}'}{\widetilde{\kappa}'\sqrt{1+\widetilde{g}^2}}$$

is a constant. From corollary 11,we obtain

$$\frac{ds_b}{ds^*} = \frac{-\epsilon c_1}{\lambda}$$

and

$$s_b = \frac{-\epsilon c_1}{\lambda}s^* + c_2$$

where c_1 and c_2 are constants, too.

We can give the following theorem to characterize a Bertrand mate curve as a slant helix.

Theorem 12 *Let $(\gamma, \widetilde{\gamma})$ be a non-helical and non-planar Bertrand pair.Thus, the Bertrand mate curve is a slant helix if and only if the binormal indicatrix of the Bertrand curve is a spherical helix.*

PROOF. Let Γ be the geodesic curvature of the principal image of the principal normal indicatrix of the Bertrand curve. From (5) and (11), we get

$$\Gamma = \frac{\tau_b}{\kappa_b}.$$

Thus, it is easy to see that the Bertrand mate curve is a slant helix if and only if the binormal indicatrix of the Bertrand curve is a spherical helix.

We can give the following corollary by using theorem 2.

Corollary 13 *A non-helical and non-planar Bertrand curve is a slant helix if and only if the binormal indicatrix of the Bertrand curve is a spherical helix.*

Thus, we can give the following theorem to characterize the spherical image of the binormal indicatrix of a Bertrand curve as a helix.

Theorem 14 *Let $(\gamma, \widetilde{\gamma})$ be a non-helical and non-planar Bertrand pair.Thus, the binormal indicatrix of the Bertrand curve is a spherical helix if and only if*

$$\widetilde{\kappa}''\widetilde{\kappa}\widetilde{f}^2 - 3\widetilde{\kappa}'^2\widetilde{g}\widetilde{f} + \widetilde{\kappa}''\widetilde{\kappa} - 3\widetilde{\kappa}'^2 = 0$$

is satisfied.

Corollary 15 *Let $(\gamma, \widetilde{\gamma})$ be a non-helical and non-planar Bertrand pair, thus, the followings are equivalent.*

i. *The tangent indicatrix of the Bertrand curve is a spherical helix.*

ii. *The principal normal indicatrix of the Bertrand curve is planar.*

iii. *The binormal indicatrix of the Bertrand curve is a spherical helix.*

The equivalence in Corollary 15 can easily prove such as $i \Rightarrow ii, ii \Rightarrow iii$ and $iii \Rightarrow i$. We can give the following corollary by using (7), (9) and (10)

Corollary 16 *The equations between the Frenet vectors of the spherical indicatrices of a non-helical and non-planar Bertrand curve in Euclidean 3-space are*

$$T_t = -\epsilon T_b,\ T_n = -\epsilon N_t = N_b \text{ and } B_t = B_b.$$

3 Spherical indicatrices of Bertrand Mate Curve

In this section, we introduced the spherical indicatrices of a Bertrand mate curve according to the Bertrand curve in Euclidean 3-space and gave consid-

erable results by using the properties of the curves, similar to the previous section. Let γ be a Bertrand curve with a Bertrand mate curve $\widetilde{\gamma}$. Then we can write the following equation by using (1)

$$\widetilde{\gamma}(s^*) = \gamma(s) + \lambda N(s) \tag{13}$$

where λ is

$$\lambda = \frac{g}{\kappa\,(g-f)} \tag{14}$$

and

$$f = \frac{\tau}{\kappa}, \quad g = \frac{\tau'}{\kappa'}.$$

We used the same λ in (1) to see the relation between (2) and (14) such that

$$\frac{g}{\kappa\,(g-f)} = \frac{-\epsilon\widetilde{g}}{\widetilde{\kappa}\left(\widetilde{g}-\widetilde{f}\right)}.$$

So, we can state that the following equation is satisfied for any non-helical and non-planar Bertrand curve and its mate curve which is non-helical and non-planar too.

$$(\widetilde{\kappa}+\epsilon\kappa)\,g\widetilde{g} - \epsilon f\widetilde{g}\kappa - \widetilde{f}g\widetilde{\kappa} = 0$$

Theorem 17 *Let $\widetilde{\gamma}$ be a Bertrand mate curve, then we have Frenet formula:*

$$\widetilde{T}' = \widetilde{\kappa}\widetilde{N}, \quad \widetilde{N}' = -\widetilde{\kappa}\widetilde{T} + \widetilde{\tau}\widetilde{B}, \quad \widetilde{B}' = -\widetilde{\tau}\widetilde{N}$$

where

$$\widetilde{T} = -\frac{1}{\sqrt{1+g^2}}\left\{T - gB\right\}, \quad \widetilde{N} = \epsilon N, \quad \widetilde{B} = -\frac{\epsilon}{\sqrt{1+g^2}}\left\{gT + B\right\}$$

and

$$ds^* = \int \frac{f\sqrt{1+g^2}}{(g-f)}ds, \quad \widetilde{\kappa} = \frac{-\epsilon\kappa\,(g-f)\,(1+fg)}{f\,(1+g^2)}, \quad \widetilde{\tau} = \frac{\kappa\,(g-f)^2}{f\,(1+g^2)}.$$

The geodesic curvature of the the principal image of the principal normal indicatrix of Bertrand mate curve is

$$\widetilde{\Gamma} = \frac{\kappa' f\,(1+g^2)^2}{-\kappa^2\left((1+fg)^2 + (g-f)^2\right)^{3/2}}\frac{ds}{ds^*}. \tag{15}$$

Let $q_1 = \frac{1}{\sqrt{1+g^2}}$, and $q_2 = \frac{g}{\sqrt{1+g^2}}$. If we differentiate the equation (13) with respect to s, we get

$$\widetilde{\gamma}'(s^*)\frac{ds^*}{ds} = \frac{-f}{(g-f)}T + \frac{fg}{(g-f)}B.$$

Since

$$\frac{ds}{ds^*} = \frac{(g-f)}{f\sqrt{1+g^2}},$$

we can write

$$\widetilde{\gamma}'(s^*) = -q_1 T + q_2 B$$

again, if we differentiate the last equation with respect to s, we obtain

$$\widetilde{\gamma}''(s^*) = -q_1' \frac{ds}{ds^*} T - \kappa \{q_1 + q_2 f\} \frac{ds}{ds^*} N + q_2' \frac{ds}{ds^*} B$$

Since $\widetilde{\gamma}''(s)$ has no components on T and B, $q_1' = 0$ and $q_2' = 0$.Hence, q_1 and q_2 are constants. Thus, we prove the following theorem.

Theorem 18 *Let γ and $\widetilde{\gamma}$ be the regular curves in E^3.If $\widetilde{\gamma}$ is the Bertrand mate curve of γ, g is a constant.*

From the theorems 3 and 18, we can conclude that there exists the following relation between g and $\widetilde{g}$ for a Bertrand pair $(\gamma, \widetilde{\gamma})$ in E^3.

$$\epsilon g + \widetilde{g} = 0.$$

Corollary 19 *Let γ and $\widetilde{\gamma}$ be regular curves in E^3. The pair of $(\gamma, \widetilde{\gamma})$ is a Bertrand pair if and only if g and $\widetilde{g}$ are constants.*

The tangent indicatrix of the Bertrand mate curve is

$$\widetilde{\gamma}_t = \frac{-1}{\sqrt{1+g^2}} \{T - gB\}.$$

Theorem 20 *If the Frenet frame of the tangent indicatrix $\widetilde{\gamma}_t = \widetilde{T}$ of a Bertrand mate curve is $\{\widetilde{T}_t, \widetilde{N}_t, \widetilde{B}_t\}$, we have Frenet formula:*

$$\widetilde{T}_t' = \widetilde{\kappa}_t \widetilde{N}_t, \quad \widetilde{N}_t' = -\widetilde{\kappa}_t \widetilde{T}_t + \widetilde{\tau}_t \widetilde{B}_t, \quad \widetilde{B}_t' = -\widetilde{\tau}_t \widetilde{N}_t$$

where

$$\widetilde{T}_t = -N \quad , \quad \widetilde{N}_t = \frac{1}{\sqrt{1+f^2}} \{T - fB\} \quad , \quad \widetilde{B}_t = \frac{1}{\sqrt{1+f^2}} \{fT + B\} \tag{16}$$

and

$$s_t^* = -\int \frac{\kappa (g-f)^2}{f(1+g^2)} ds^*, \quad \widetilde{\kappa}_t = \frac{\sqrt{1+f^2}\sqrt{1+g^2}}{(f-g)}, \quad \widetilde{\tau}_t = \frac{\kappa' \sqrt{1+g^2}}{\kappa^2 (1+f^2)} \tag{17}$$

and s_t^* is a natural representation of the tangent indicatrix of the curve $\widetilde{\gamma}$. $\widetilde{\kappa}_t$ and $\widetilde{\tau}_t$ are the curvature and torsion of $\widetilde{\gamma}_t$, respectively. The geodesic curvature

of the principal image of the principal normal indicatrix of $\widetilde{\gamma}_t$ is

$$\widetilde{\Gamma}_t = .\frac{\kappa^3 (f-g)^2 (1+f^2)^{3/2} \left\{ (\kappa\kappa'' - 3\kappa'^2)(1+f^2) - 3f\kappa'^2 (g-f) \right\}}{\sqrt{1+g^2}\left(\kappa^4 (1+f^2)^3 + \kappa'^2 (f-g)^2\right)^{3/2}} \frac{ds}{ds_t^*}. \tag{18}$$

Theorem 21 *Let $(\gamma, \widetilde{\gamma})$ be a non-helical and non-planar Bertrand pair parameterized by the arclengths s and s^*. Thus, the followings are true.*

i. Bertrand mate curve is a slant helix if and only if the tangent indicatrix of the Bertrand mate curve is a spherical helix.

ii. Bertrand curve is a slant helix if and only if the tangent indicatrix of the Bertrand mate curve is a spherical helix.

iii. Bertrand curve is a slant helix if and only if the tangent indicatrix of the Bertrand curve is a spherical helix.

The arclengths s_t^* and s satisfy the equation

$$s_t^* = \int \frac{\kappa (f-g)}{\sqrt{1+g^2}} ds.$$

We can re-state theorem 6 for Bertrand mate curve to characterize its spherical image of tangent indicatrix as a helix without using " $\sim$ ".

The principal normal indicatrix of the Bertrand mate curve is

$$\widetilde{\gamma}_n = \epsilon N.$$

Theorem 22 *If the Frenet frame of the principal normal indicatrix $\widetilde{\gamma}_n = \widetilde{N}$ of a non-helical and non-planar Bertrand mate curve is $\{\widetilde{T}_n, \widetilde{N}_n, \widetilde{B}_n\}$, we have Frenet formula:*

$$\widetilde{T}_n' = \widetilde{\kappa}_n \widetilde{N}_n, \quad \widetilde{N}_n' = -\widetilde{\kappa}_n \widetilde{T}_n + \widetilde{\tau}_n \widetilde{B}_n, \quad \widetilde{B}_n' = -\widetilde{\tau}_n \widetilde{N}_n$$

where

$$\begin{aligned}
\widetilde{T}_n &= \frac{-\epsilon}{\sqrt{1+f^2}} \{T - fB\} \\
\widetilde{N}_n &= \frac{\epsilon}{\sigma\sqrt{1+f^2}} \left\{ f\kappa'(g-f)T - \kappa^2 \left(1+f^2\right)^2 N + \kappa'(g-f)B \right\} \\
\widetilde{B}_n &= \frac{1}{\sigma} \left\{ f\kappa^2 \left(1+f^2\right) T + \kappa'(g-f)N + \kappa^2 \left(1+f^2\right) B \right\}
\end{aligned} \tag{19}$$

where

$$\sigma = \sqrt{\kappa'^2 \left(g-f\right)^2 + \kappa^4 \left(1+f^2\right)^3}$$

and

$$s_n^* = \int \frac{\kappa \left(g-f\right)\sqrt{1+f^2}}{f\sqrt{1+g^2}} ds^*, \quad \widetilde{\kappa}_n = \frac{\sigma}{\kappa^2 \left(1+f^2\right)^{3/2}}$$
$$\widetilde{\tau}_n = \frac{-\epsilon(g-f)}{\sigma^2} \left\{3f\kappa'^2 \left(g-f\right) + \left(3\kappa'^2 - \kappa\kappa''\right)\left(1+f^2\right)\right\}$$

and s_n^* is a natural representation of the principal normal indicatrix of the curve $\widetilde{\gamma}$. $\widetilde{\kappa}_n$ and $\widetilde{\tau}_n$ are the curvature and torsion of $\widetilde{\gamma}_n$, respectively.

The arclengths s_n^* and s satisfy equation

$$s_n^* = \int \kappa\sqrt{1+f^2} ds.$$

We can re-state theorem 8 for Bertrand mate curve to characterize its spherical image of principal indicatrix as a planar curve without using " $\sim$ ".

The binormal indicatrix of the Bertrand mate curve is

$$\widetilde{\gamma}_b = \frac{-\epsilon}{\sqrt{1+g^2}} \left\{gT + B\right\}.$$

If the Frenet frame of the binormal indicatrix $\widetilde{\gamma}_b = \widetilde{B}$ of a Bertrand mate curve is $\{\widetilde{T}_b, \widetilde{N}_b, \widetilde{B}_b\}$, we have Frenet formula:

$$\widetilde{T}_b' = \widetilde{\kappa}_b \widetilde{N}_b, \quad \widetilde{N}_b' = -\widetilde{\kappa}_b \widetilde{T}_b + \widetilde{\tau}_b \widetilde{B}_b, \quad \widetilde{B}_b' = -\widetilde{\tau}_b \widetilde{N}_b$$

where

$$\widetilde{T}_b = \epsilon N, \;\; \widetilde{N}_b = \frac{-\epsilon}{\sqrt{1+f^2}} \left\{T - fB\right\}, \;\; \widetilde{B}_b = \frac{1}{\sqrt{1+f^2}} \left\{fT + B\right\} \tag{20}$$

and

$$s_b^* = -\int \frac{\kappa \left(f-g\right)^2}{f\left(1+g^2\right)} ds^*, \quad \widetilde{\kappa}_b = \frac{\sqrt{1+g^2}\sqrt{1+f^2}}{\left(f-g\right)}, \quad \widetilde{\tau}_b = \frac{-\epsilon\kappa'\sqrt{1+g^2}}{\kappa^2\left(1+f^2\right)} \tag{21}$$

and s_b^* is a natural representation of the binormal indicatrix of the curve $\widetilde{\gamma}$ and it is equal to the total torsion of the curve $\widetilde{\gamma}$. $\widetilde{\kappa}_b$ and $\widetilde{\tau}_b$ are the curvature and torsion of $\widetilde{\gamma}_b$, respectively. The geodesic curvature of the principal image of the principal normal indicatrix of $\widetilde{\gamma}_b$ is

$$\widetilde{\Gamma}_b = \frac{\kappa^3 \left(f-g\right)^2 \left(1+f^2\right)^{3/2} \left\{\left(\kappa\kappa'' - 3\kappa'^2\right)\left(1+f^2\right) - 3f\kappa'^2\left(g-f\right)\right\}}{\sqrt{1+g^2}\left(\kappa^4\left(1+f^2\right)^3 + \kappa'^2\left(f-g\right)^2\right)^{3/2}} \frac{ds}{ds_b^*}. \tag{22}$$

From equations (17), (18), (21) and (22), we conclude that

$$\frac{\widetilde{\tau}_t}{\widetilde{\kappa}_t} = \frac{\widetilde{\tau}_b}{\widetilde{\kappa}_b}$$

and $\widetilde{\Gamma}_t = \widetilde{\Gamma}_b$.

Corollary 23 *The arclengths s_b^* and s satisfy the following equation.*

$$s_b^* = \int \frac{\kappa\,(f-g)}{\sqrt{1+g^2}} ds$$

Since λ is a constant, from (14), we can write

$$\tau' = \lambda\kappa\kappa'(g-f)$$

and since q_2 is a constant, we can write

$$\tau' = c_1\kappa'\sqrt{1+g^2}.$$

Thus, we obtain

$$\frac{\kappa(g-f)}{\sqrt{1+g^2}} = \frac{c_1}{\lambda}$$

and we can also say that

$$\frac{\kappa(g-f)}{\sqrt{1+g^2}}$$

is a constant. From corollary 23, we obtain

$$\frac{ds_b^*}{ds} = \frac{c_1}{\lambda}$$

and

$$s_b^* = \frac{c_1}{\lambda}s + c_2$$

where c_1 and c_2 are constants, too.

Theorem 24 *Let $(\gamma, \widetilde{\gamma})$ be a non-helical and non-planar Bertrand pairs parameterized by arclengths s and s^*. Thus, the Bertrand curve is a slant helix if and only if the binormal indicatrix of the Bertrand mate curve is a spherical helix.*

The theorem 24 can easily prove as similar to theorem 12. Furthermore, we can re-state theorem 14 for Bertrand mate curve to characterize its spherical image of the binormal indicatrix as a spherical helix without using " $\sim$ " and we can re-state corollary 15 for Bertrand mate curves.

We can give the following corollary by using (16), (19), and (20).

Corollary 25 *The equations between the Frenet vectors of the spherical indicatrices of a Bertrand mate curve in Euclidean 3-space are*

$$\widetilde{T}_t = -\epsilon \widetilde{T}_b, \ \widetilde{N}_t = -\epsilon \widetilde{T}_n = -\epsilon \widetilde{N}_b \ \textit{and} \ \widetilde{B}_t = \widetilde{B}_b.$$

Both the spherical indicatrices of Bertrand curve and Bertrand mate curve were examined in detail and it was seen that none of them didn't create any specific curve pairs as a mutual.

Acknowledgement

We would like to thank the referee for valuable suggestions which improved the paper considerably.

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